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Almost all devices in the modern world are based on charged particles in motion. Charged particles are influenced by the electromagnetic force, which alters their motion. An analogy is a roller coaster in an amusement park. We know that an object with a mass is accelerated by the gravitational force. In the same way, an object with a charge is accelerated by an electromagnetic force. In a picture tube, the electrons

that emerge from the heated cathode are accelerated by an electrostatic force and then deflected by a magnetic force.

## 2.2 Basic electrostatic phenomenon

Fig.2.2.1 shows that charge (electrons) can be transferred from one object to another, when we rub plastic rods against fur or glass rods against silk. Charged objects with the same algebraic sign repel. Charged objects with opposite algebraic sign attract.

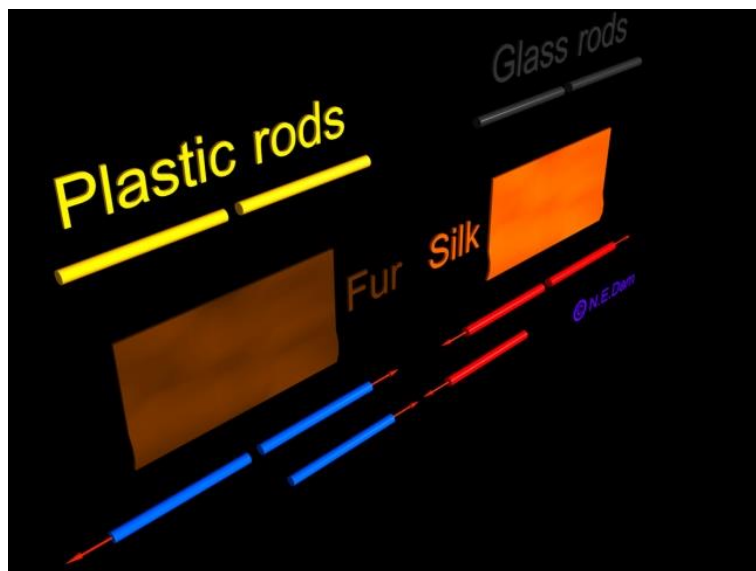


Fig.2.2.1, (N.E. Dam ©). Basic electrostatic phenomenon.

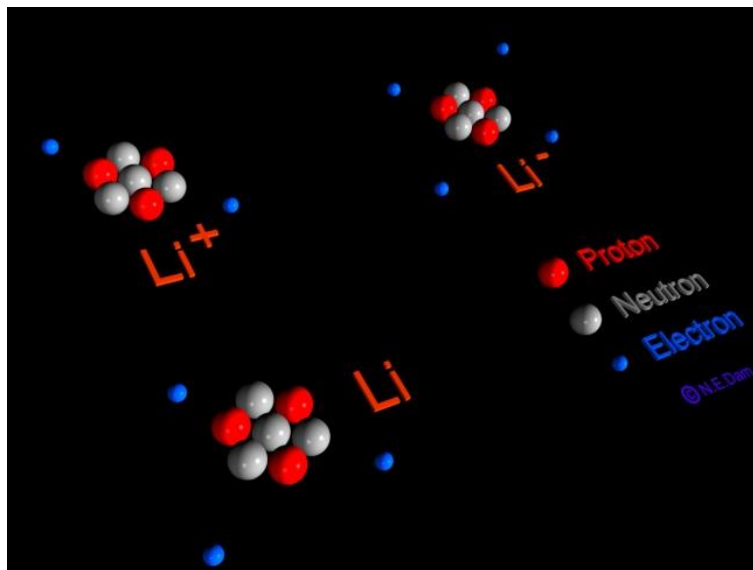


Fig.2.2.2, (N.E. Dam ©). A neutral Li (lithium) atom, a positive Li ion and a negative Li ion. The size of the whole atom including electrons is about 0,1 nm, the size of the nucleus is about 1 fm. The number of protons is called the atomic number.

The electric charge is quantized and obeys a conservation principle. If the charges are in rest, we are dealing with electrostatic interactions.

Electrostatic interactions are described by Coulomb's law and electric fields. An atom consists of a nucleus (positively charged protons and uncharged neutrons held together by strong nuclear forces) and negatively charged electrons, that "circle around" the core, see fig.2.2.2. In a neutral atom, the number of electrons equals the number of protons. If the number of electrons is changed, the atom is ionized. If electrons are gained, the atom becomes a negative ion. If electrons are lost, the atom becomes a positive ion.

In the same way, a neutral body has an equal number of positive and negative charges. If electrons are gained, the body becomes negatively charged. If electrons are lost, the body becomes positively charged.

In a closed system, the charge cannot disappear. For example, when the plastic rod is rubbed with a piece of fur, electrons are transferred from the fur to the rod. The fur becomes positively charged and the plastic rod becomes negatively charged.

If "we" remove one electron from a metal atom, which is situated in a crystal lattice, the positive ion will remain fixed at that location in the lattice, while the electron is free to move around in the whole crystal lattice (the whole bulk of metal).

The charge  $q$  is always an integer multiple  $n$  of the charge quantum

$$e = 1,602 \times 10^{-19} \text{ C} \quad \text{eq.2.2.1}$$

$$q = n e \quad \text{eq.2.2.2}$$

The charge of an electron is  $q = -1 e = -e$ .

## 2.3 How to charge an object

### Charging by contact

Fig.2.3.1 shows negative charge transferred from a plastic rod via a copper wire to a metal sphere.

In fig.2.3.2 we have removed the copper wire. Now, the negatively charged rod repels the negatively charged metal sphere.

In fig.2.3.3 we have replaced the negatively charged rod by a positively charged rod. The positively charged rod attracts the negatively charged metal sphere.

Conductors permit easy motion of charge, while insulators do not.

Normally, metals are good conductors, while non-metals are good insulators. In a metal, the mobile charge is negative electrons. In ionic solutions and in ionized gases, the mobile charge may be positive and negative at the same time.

Metals are in fact metal crystals, where one outer electron from each atom is detached from the atom and allowed to move freely around in the crystal lattice. Such a free electron is called a conduction electron. Each atom contributes with one electron, so there are as many free (conduction) electrons as there are atoms. In some metals there may be released more than one conduction electron from each atom. The free (conduction) electrons transport the charge. The other electrons remain bound to the nucleus.

Inside insulators all the electrons are bound to the nucleus and can only move a short distance within an atom or within a molecule.

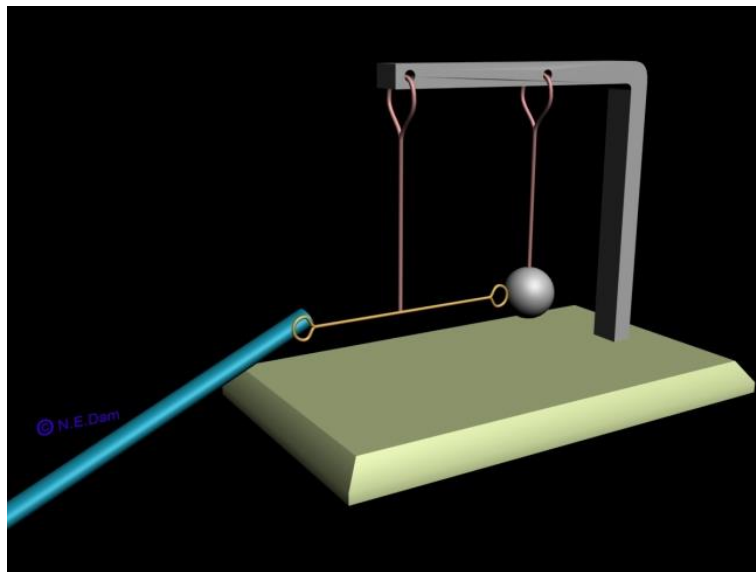


*Fig.2.3.1 (N.E. Dam ©). Charging by contact. Negative charge is transferred from a plastic rod via a copper wire to a metal sphere.*

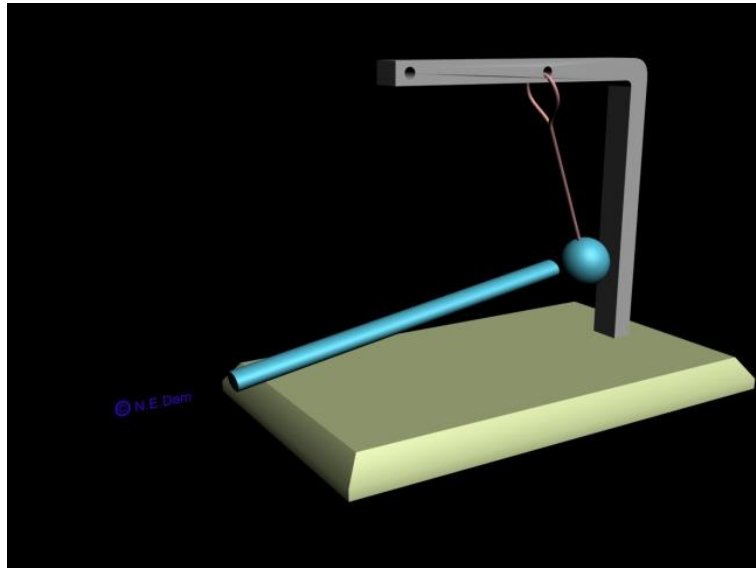


*Fig.2.3.2, (N.E. Dam ©). Charging by contact. The negatively charged rod repels the negatively charged metal sphere.*

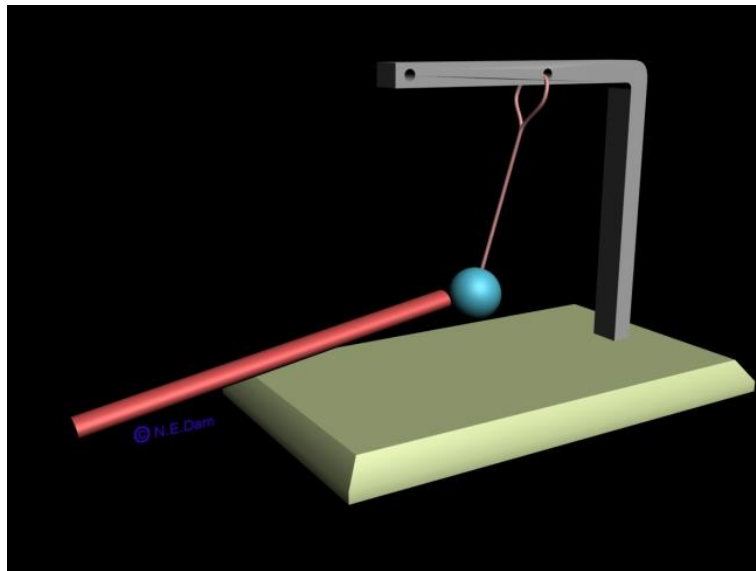


*Fig.2.3.3, (N.E. Dam ©). Charging by contact. A positively charged rod attracts the negatively charged metal sphere.*

### Charging by induction

In fig.2.3.1 the metal sphere was charged by direct contact. The metal sphere may also be charged in another way as shown in fig.2.3.4-7, where the metal sphere is charged by induction.

Fig.2.3.4. The rod may be the plastic rod from fig.2.3.1. It has an excess of negative charge, which repels electrons in the right part of the surface of the metal sphere. These electrons move to the left side of the surface of the metal sphere. After some time, there are so many electrons on the left side, that it impedes further migration of electrons from the right side to the left side of the metal sphere. An equilibrium stabilizes the situation. The plastic rod exerts a net attractive force on the metal sphere, even though the total charge on the sphere is zero, because the positive charges are closer to the rod than the negative charges.

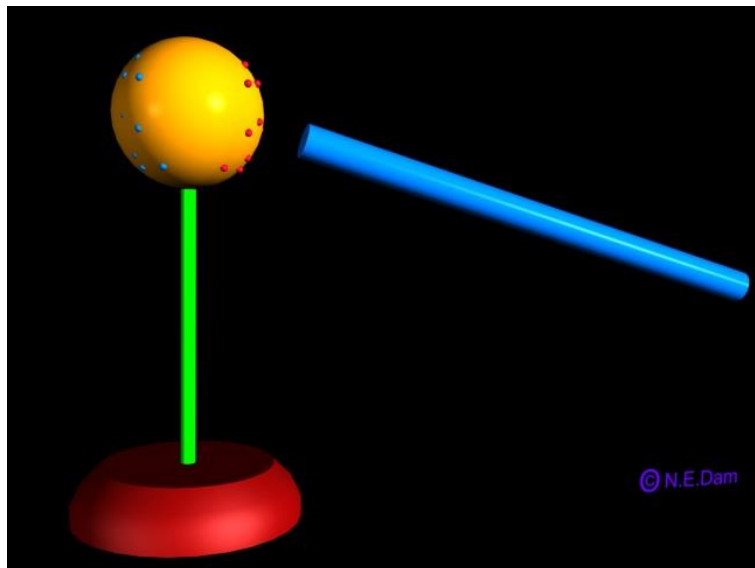


*Fig.2.3.4, (N.E. Dam ©). Charging by induction.  
The metal sphere is charged by induction from the rod.*

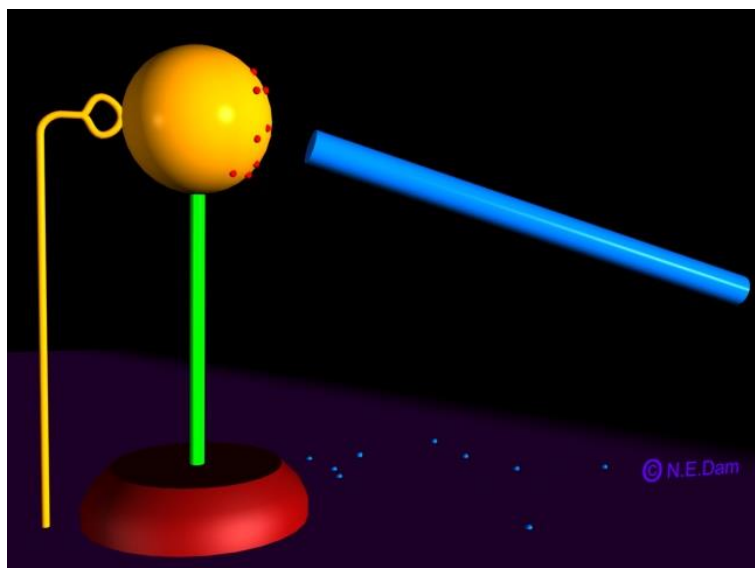


*Fig.2.3.5, (N.E. Dam ©). Charging by induction.  
The excess electrons flow down to the ground.*

Fig.2.3.5. If a conductor touches the left surface of the metal sphere, the excess electrons are transferred to the ground.

Fig.2.3.6. If the conductor is then removed, the positive charge on the right surface of the sphere is fixed because of the presence of the negative plastic rod.

Fig.2.3.7. If the plastic rod is removed, the positive charge is uniformly distributed on the whole surface of the sphere.

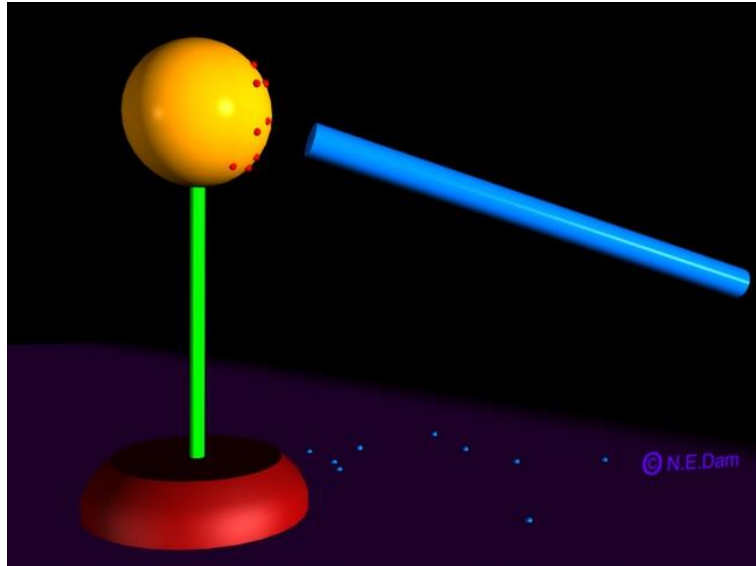


*Fig.2.3.6, (N.E. Dam ©). Charging by induction.  
When the excess electrons are removed.*

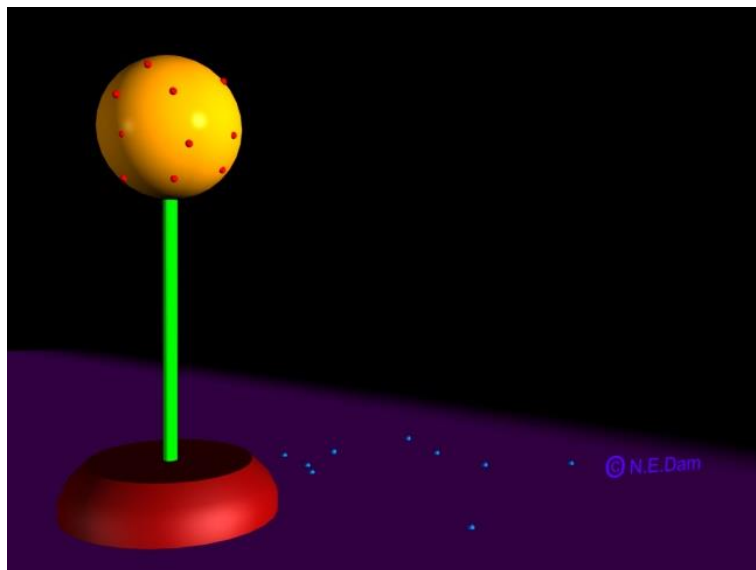


*Fig.2.3.7, (N.E. Dam ©). Charging by induction.  
When the rod is removed.*

## Polarization

Fig.2.3.8 shows the attraction between a comb and some pieces of plastic or paper. In the left side of the figure, a negatively charged plastic comb (blue) causes a slight shifting of charge within the molecules of the neutral insulator (green), an effect called polarization. Hereby the positive charges in the neutral insulator are closer to comb than the negative charges.

Consequently, the attractive forces between the comb and the positive charges in the insulator will be greater than the repulsive forces between the comb and the negative charges in the insulator.

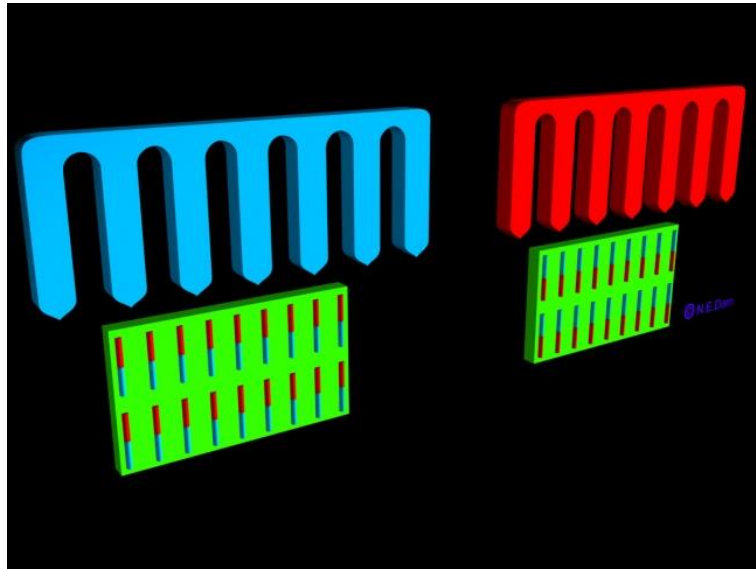


*Fig.2.3.8, (N.E. Dam ©). Polarization of an insulator.*

The result is a net force between the negative charged comb and the neutral insulator. Same argumentation may be used in the right side of the figure, where the comb is positively charged (red).

Note the similarity and the difference relative to the charging by induction explained above for a metal.

## 2.4 The electrostatic force between charges

### Coulomb's law for two charges

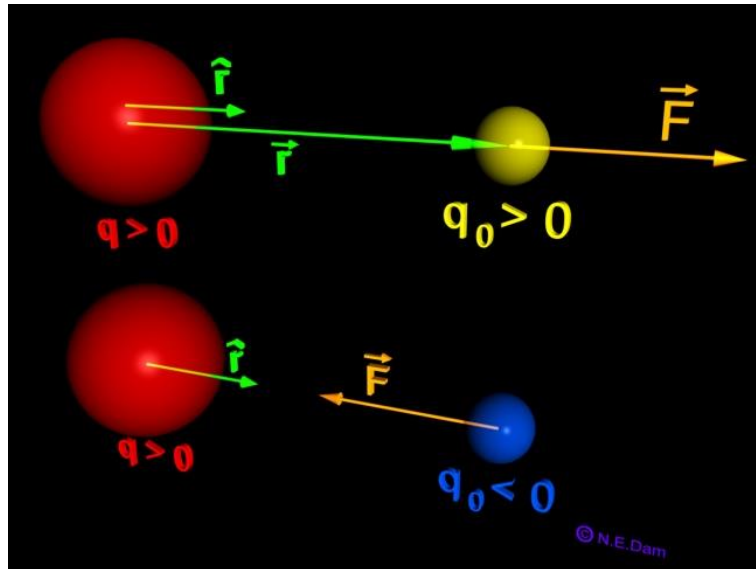


Fig.2.4.1, (N.E. Dam ©). Coulomb's law. Interaction between  $q$  and  $q_0$ . Only the force on  $q_0$  is shown.

Coulomb (1736-1806) derived an expression for the force between charged particles. For two charges  $q$  and  $q_0$ , as shown in fig.2.4.1, he found that the force is proportional to both charges and inversely proportional to the squared distance  $r^2$  between the charges

$$F_e = k_e \frac{|q \cdot q_0|}{r^2} \quad \text{eq. 2.4.1}$$

Subscript “e” refers to “electric”. The proportionality constant is defined as

$$k_e = \frac{1}{4\pi\epsilon_0} = 8,99 \cdot 10^9 \frac{\text{Nm}^2}{\text{C}^2} \quad \text{eq. 2.4.2}$$

Extra info (not important)

The constant  $\epsilon_0$  is the so-called vacuum permittivity

$$\epsilon_0 = 8,85 \cdot 10^{-12} \frac{\text{C}^2}{\text{Nm}^2} \quad \text{eq. 2.4.3}$$

It is related to  $c$ , the speed of light in vacuum, and to the so-called vacuum permeability

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{\text{Ns}^2}{\text{C}^2} \quad \text{eq. 2.4.4}$$

through the equation

$$c^2 = \frac{1}{\epsilon_0 \mu_0} \quad \text{eq. 2.4.5}$$

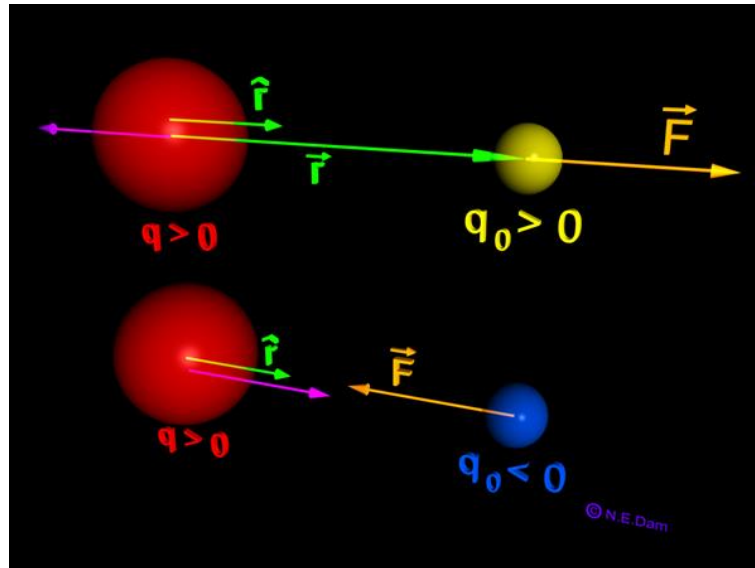


Fig.2.4.2, (N.E. Dam ©). Coulomb's law. The force on  $q_0$  and the force on  $q$  is shown. The two forces are of the same magnitude, but opposite directed according to Newton's third law.

In fig.2.4.1 and in fig.2.4.2 we have defined the distance from  $q$  to  $q_0$  as vector  $\vec{r}$ .

*The length of vector  $\vec{r}$  is the distance  $r$  and the direction of  $\vec{r}$  is the direction from  $q$  to  $q_0$ !*

The unit vector of  $\vec{r}$  is defined as

$$\hat{r} \equiv \frac{\vec{r}}{r} \quad \text{eq. 2.4.6}$$

It shows the direction from  $q$  to  $q_0$ . Now we can write

### Coulomb's law in vector form

$$\vec{F}_e = k_e \frac{q \cdot q_0}{r^2} \hat{r} \quad \text{eq. 2.4.7}$$

This is the force that  $q$  exerts on  $q_0$ .

We have no absolute sign in the numerator. If the two charges have opposite sign, the numerator becomes negative and the force on  $q_0$  will be in the opposite direction of  $\hat{r}$ .

It should be noted, that gravitational forces for charged bodies are normally negligible in comparison to the electrostatic forces.

As mentioned, the charge is quantized. The quantum is the charge on a proton or minus the charge on an electron,  $e = 1,602 \cdot 10^{-19}$  C. The unit of electric charge is Coulomb. It is defined through one Ampere (which is 1 Coulomb per second).

### How Coulomb measured the electrostatic force

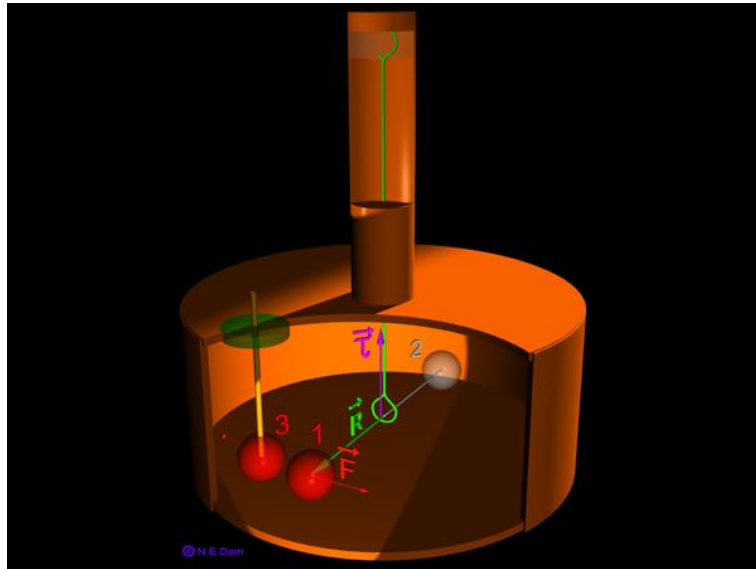


*Fig.2.4.3, (N.E. Dam ©). The principle of a torsion balance. The instrument is cut through to see what is inside.*

#### Explanation of fig.2.4.3-4

Two spheres 1 and 2 are connected by an insulating beam and suspended in a thin filament, eg. a silk thread. Sphere 2 compensates for the weight of sphere 1. Sphere 1 and 3 are both covered with metal. Sphere 3 is fixed to a conducting rod. The rod is fixed to the instrument and insulated from the instrument. To begin with, the two spheres 1 and 3 touch each other. Hence they will get the same charge, when charge is transferred to sphere 3 through the fixed rod.

The original set-up was invented by Coulomb in 1777 and independently by Michell before 1783. The instrument was used by Coulomb to measure the electrostatic force between charges, and by Cavendish in 1798 to measure the gravitational force between two masses.

Note, that the distance  $r$  used in Coulomb's law is the distance between the centres of the two charges (sphere 1 and 3). The two spheres (1 and 3) are small compared with the distance between them. Note, that  $R$  is the radius from the filament to sphere 1 used to calculate the torque.

Coulomb did a lot of experiments and found, that the force is proportional to both charges and inversely proportional to the squared distance  $r^2$  between the charges.

In fig.2.4.3 the repulsive force  $F$  between the two charged spheres (1 and 3) will give a *torque* on the rod. In general the torque vector is defined as

$$\vec{\tau} = \vec{R} \times \vec{F} \quad \text{eq. 2.4.8}$$

In this case

$$\vec{R} \perp \vec{F} \Rightarrow \tau = R F \quad \text{eq. 2.4.9}$$

If a torsion spring, filament or thread is twisted through the angle  $\theta$ , the filament will try to rotate back again, and the twisting torque will be proportional to  $\theta$

$$\tau = \kappa \theta \quad \text{eq. 2.4.10}$$

where  $\kappa$  is the spring constant for the torsion filament<sup>1</sup>.

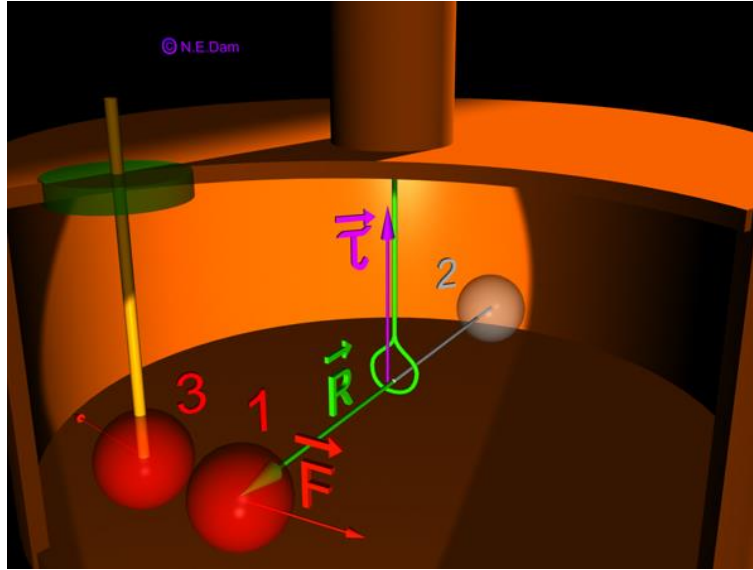


Fig.2.4.4, (N.E. Dam ©). Close-up of the torsion balance.

In fig.2.4.4 the twisting torque from the filament will act in the opposite direction of the torque from the repulsion force acting on sphere 1. In balance, the magnitude of the two torques equals:

$$R F = \kappa \theta \Rightarrow F = \frac{\kappa \theta}{R} \quad \text{eq. 2.4.11}$$

From this  $F$  may be found, since  $R$  and  $\theta$  can be measured and  $\kappa$  may be found experimentally.

### The total force if there are more than two charges

*The principle of superposition* is used to calculate the total force on  $q_0$ , if more than two charges are present. So far we have calculated the force that  $q$  exerts on  $q_0$ , i.e. the force on  $q_0$ , when  $q$  is the only other charge present. In the case when there are two other charges  $q_1$  and  $q_2$  instead of just  $q$ , the total force on  $q_0$  is the sum of the following two forces :

1. The force  $\vec{F}_1$  that  $q_1$  exerts on  $q_0$ , when  $q_2$  is not present.
2. The force  $\vec{F}_2$  that  $q_2$  exerts on  $q_0$ , when  $q_1$  is not present.

The total electric force on  $q_0$  is then the sum

$$\vec{F}_e = \vec{F}_1 + \vec{F}_2 \quad \text{eq. 2.4.12}$$

<sup>1</sup> To find  $\kappa$  the torsion balance should be free. Then the torsion balance should be twisted and released, and the period of oscillation time should be measured. This period depends on the moment of inertia of the beam and the elasticity of the filament. Since the the moment of inertia of the beam can be found from its mass, the torsion constant  $\kappa$  can be calculated.

This is the principle of superposition. It has been verified experimentally. The principle may be expanded to more charges. The total force on  $q_0$  is the sum of the following forces :

1. The force  $\vec{F}_1$  that  $q_1$  exerts on  $q_0$  , when only  $q_1$  (and  $q_0$ ) are present.
2. The force  $\vec{F}_2$  that  $q_2$  exerts on  $q_0$  , when only  $q_2$  (and  $q_0$ ) are present...
- ..N. The force  $\vec{F}_N$  that  $q_N$  exerts on  $q_0$  , when only  $q_N$  (and  $q_0$ ) are present.

The total electric force on  $q_0$  is then the sum

$$\vec{F}_e = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N \quad \text{eq. 2.4.13}$$

The principle of superposition is illustrated by the problems 2.1 and 2.2.

## 2.5 The E-field

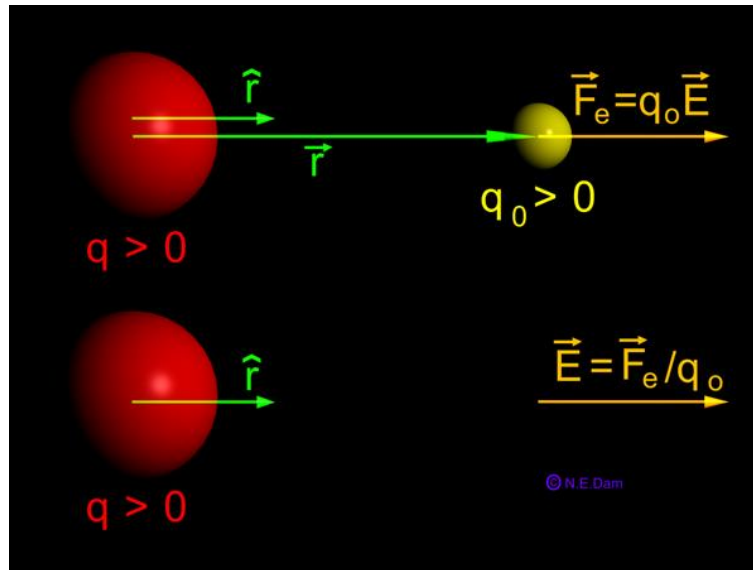


Fig.2.5.1, (N.E. Dam ©). Definition of the electric field.

### Definition of the electric field

We now define the electric field at a point P. We place a so-called test charge  $q_0$  at point P. The electric field at point P is then defined as the electric force  $\vec{F}_e$  on  $q_0$  divided by  $q_0$

$$\vec{E} \equiv \frac{\vec{F}_e}{q_0} \quad \text{eq. 2.5.1}$$

The unit is N/C. We can use eq.2.5.1 to calculate the force on  $q_0$ , if we know the E-field:

$$\vec{F}_e \equiv q_0 \vec{E} \quad \text{eq. 2.5.2}$$

### Explanation

Fig.2.5.1 shows two charged particles  $q$  and  $q_0$ . The position of  $q_0$  is the so-called field point P. According to Coulomb's law and Newton's third law,  $q$  and  $q_0$  will exert the same force on each other, but in opposite directions. The electric field at point P is a measure of how strong the force would be, if we place a charge  $q_0$  at point P. According to Coulomb's law, this force will be proportional to the magnitude of  $q_0$ , but if we divide the force by  $q_0$ , the force will be independent of  $q_0$ . Then the force only depends on the charge  $q$  and on the distance  $r$  from  $q$  to P. The force will, of course, only exist if we have a charge  $q_0$  at point P.

However, the field will exist no matter if we have a charge at point P or not, since the field is the force (divided by  $q_0$ ), that would be on  $q_0$  if we place  $q_0$  at P. Therefore  $q_0$  is called a test charge. It tests, if there is a field at the so-called field point P.

Note that the electric force on a charge is exerted by the electric field created by other charges and not by the charge itself, since a charge cannot exert a force on itself.

### Remarks

- $\vec{E}$  is the field produced by other charges than  $q_0$  and  $q_0$  is placed at point P, see fig.2.5.1.
- $q_0$  should be so small, that it does not disturb the field at its position<sup>2</sup>. Strictly, the definition should be like this:

$$\vec{E} \equiv \lim_{q_0 \rightarrow 0} \frac{\vec{F}_e}{q_0} \quad \text{eq. 2.5.3}$$

- $q_0$  should be a point charge, ie the size of  $q_0$  should – theoretically - be zero, since the E field will be different if we move a little bit away from P and we want to determine the E-field at point P and not a little bit away from P.
- If  $q_0$  is positive,  $\vec{E}$  and  $\vec{F}_e$  are in the same direction.
- eq.2.5.2 is analogous to  $\vec{F}_g = m \vec{g}$  for a mass m in a gravitational field. However, m can only be positive, while  $q_0$  may be either positive or negative. The earth sets up a gravitational g-field in the space around it, where

$$\vec{g} = \frac{\vec{F}_g}{m}$$

### The E-field from a point charge q

Fig.2.5.1 shows a point charge q. It produces an electric field and is called the source. The location of q is called the source point S. The location P is the so-called field point, where we want to determine the E-field.

We place a small test charge  $q_0$  at the field point P. According to Coulomb's law eq.2.4.7 and the definition of the electric field eq.2.5.1, we have

$$\vec{E} \equiv \frac{\vec{F}_e}{q_0} = \frac{k_e \frac{q \cdot q_0}{r^2} \hat{r}}{q_0} = k_e \frac{q}{r^2} \hat{r} \quad \text{eq. 2.5.4}$$

The field always points away from positive charge and towards negative charge, and the magnitude decreases as  $1/r^2$ , see fig.2.6.1.

At each point in space, there will be an E vector. The direction and magnitude of this vector change in general from one point to another, i.e. the three coordinates of the E-field vector depend on the position, E is a so-called vector field:

$$\vec{E} = \begin{pmatrix} E_x(x, y, z) \\ E_y(x, y, z) \\ E_z(x, y, z) \end{pmatrix} \quad \text{eq. 2.5.5}$$

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<sup>2</sup> The reason why  $q_0$  should be relatively small. If  $q_0$  is not relatively small, it affects the distribution of the charge q. E.g., if q is placed on a metal sphere as shown in fig.2.3.4, and if  $q_0$  is in the neighbourhood of q as shown in the upper part of fig.2.5.1, the charge q can redistribute on the surface of the metal sphere. The E-field, we want to determine at point P, depends on the distribution of the charge q on this metal surface. Hence,  $q_0$  should be so small, that it does not affect the charge distribution of q on the metal surface.

## 2.6 The E-field lines

The concept of field lines was introduced by Michael Faraday (1791-1867). It is an imaginary curve where the curve tangent is the direction of the electric field vector. The field lines never intersect and the density of field lines is proportional to the magnitude of the electric field.

Fig.2.6.1 shows, that the E-field lines from a single charge  $q$  point radially away from the charge and never intersect. The distance between the E-field lines increases with distance  $r$  from the centre of the charge. At the same time, the E-field becomes weaker.

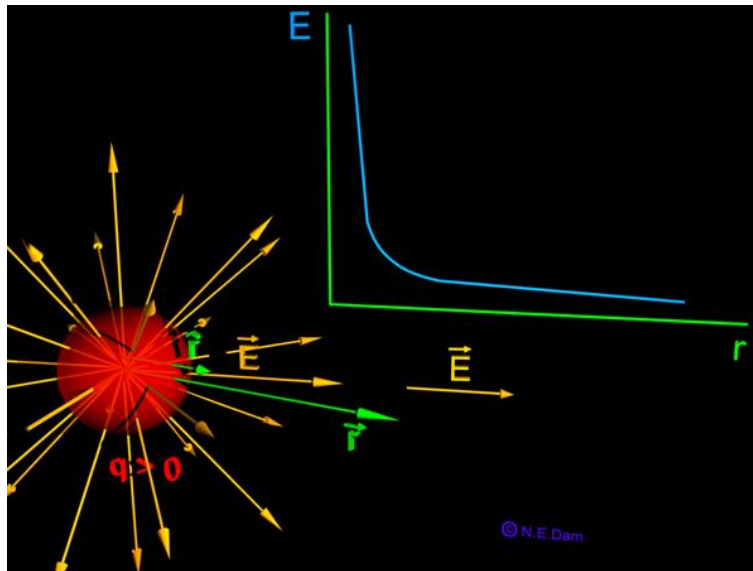


Fig.2.6.1, (N.E. Dam ©). The E-field from a single charge.

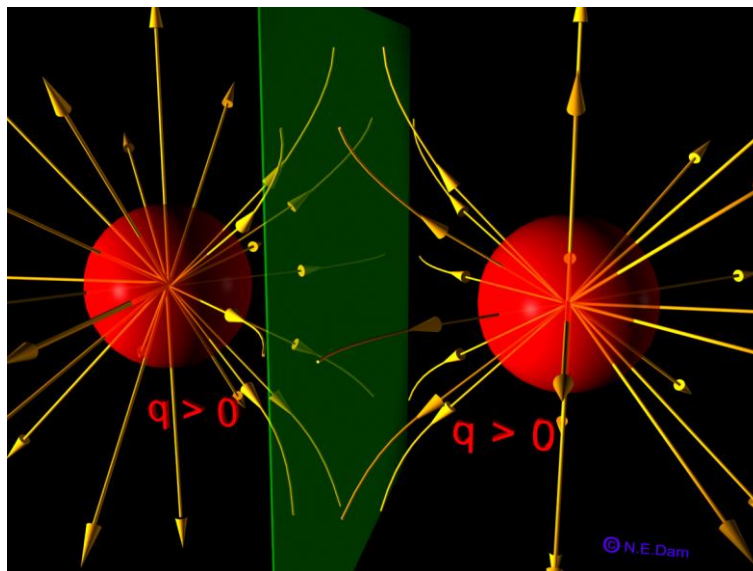


Fig.2.6.2, (N.E. Dam ©). The E-field from two positive charges.

Fig.2.6.2 shows two positive charges. We see that the E-field lines never intersect. They approach the green plane of symmetry.

Fig.2.6.3 shows a positive and a negative charge. The E-field lines all start on the positive charge and end on the negative charge as indicated by the small arrows along the E-field lines. The magnitude of the E-field is not constant along a field line. Close to one of the charges, we have a high density of E-field lines and hence a high magnitude of the E-field. In between the charges the E-field lines are more spaced, and here we have a weaker E-field (i.e. a lower magnitude or shorter E-field vector). As we see, the E-field vector always points in direction of the tangent at any point on a E-field line. The direction of the E-field vector follows the arrows on the E-field line.

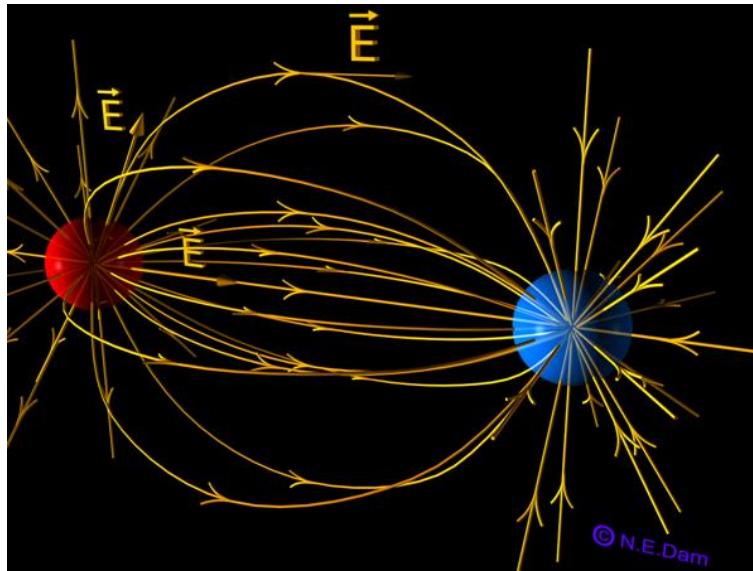


Fig.2.6.3, (N.E. Dam ©). The E-field from a positive and a negative charge of equal size (a so-called dipole).

### Calculation of E-field in general

The E-field arises from a given charge distribution.

The E-field at a point P may be calculated in two ways:

1. We calculate the total force on a test charge  $q_0$  placed at the field point P. The E-field is then calculated as total force divided by  $q_0$ .
2. We calculate the E-field  $E_1$  from  $q_1$  at point P (when  $q_2$  and other charges than  $q_1$  are "turned off"). Then we calculate the E-field  $E_2$  from  $q_2$  at point P (when other charges than  $q_2$  are "turned off") and so on. Finally we calculate the total E-field by adding all the E-fields from all the charges  $q_1, q_2, \dots$

$$\vec{E}_{\text{total}} = \vec{E} = \vec{E}_1 + \vec{E}_2 + \dots \quad \text{eq. 2.6.1}$$

Both methods are based on the principle of superposition.

Now, it is recommended to solve problem 2.3.

## 2.7 The E-field from a line of charge

The rod in fig.2.7.1 might represent one of the rods in fig.2.2.1. We want to calculate the E-field at a point P on the x-axis. We assume that the rod's charge Q is positive and uniformly distributed along the rod's length  $L=2a$ .

The linear charge density is  $\lambda = Q/L$ , and the charge in the small element  $dy$  is

$$dQ = \lambda dy = Q dy / L.$$

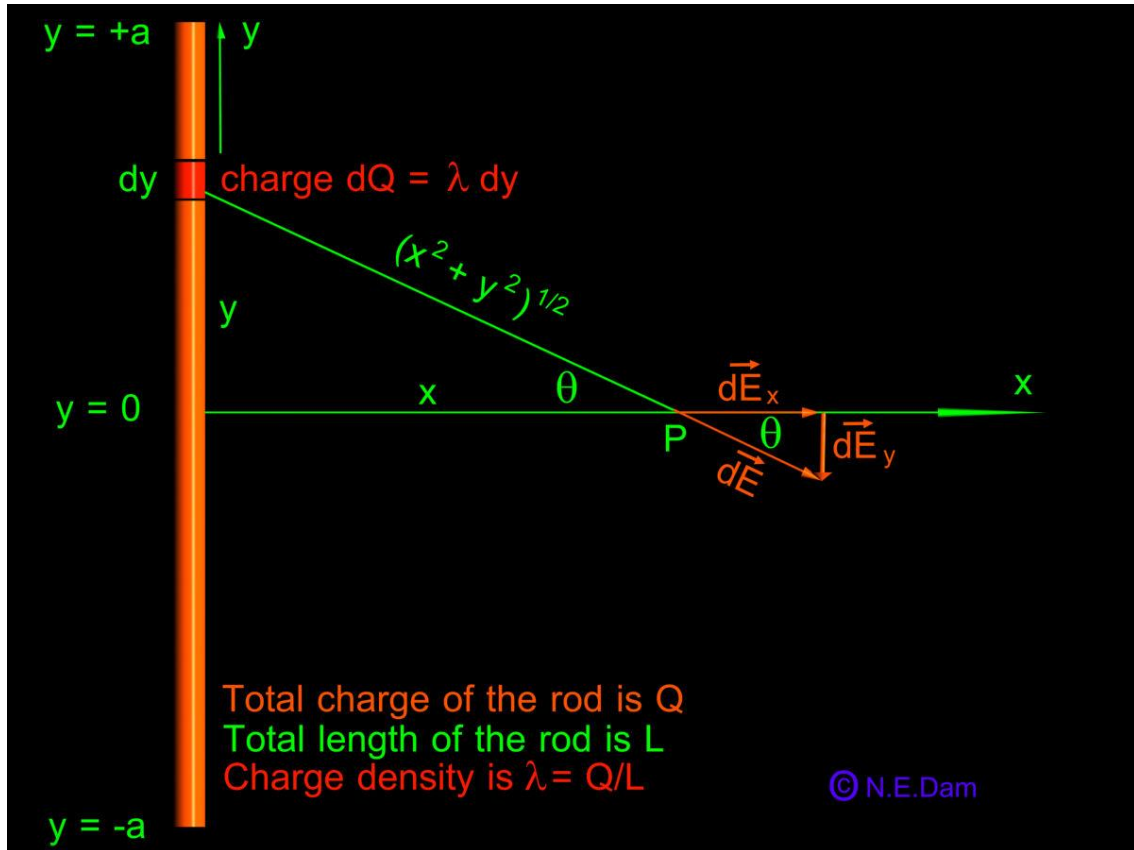


Fig.2.7.1, (N.E. Dam ©). Calculation of the E-field from a line of charge.

The field from  $dQ$  is shown in fig.2.7.1. The magnitude is

$$dE = \frac{dQ}{4\pi\epsilon_0(x^2 + y^2)} = \frac{Q dy}{8a\pi\epsilon_0(x^2 + y^2)}$$

From the figure it might be seen that the  $dE_y$  component from a charge element at position  $y$  will cancel with the  $dE_y$  component from a charge element at the position  $-y$ . Therefore, the total  $y$  component for the E-field becomes  $E_y = 0$ .

Now we will try to find total  $x$  component for the E-field. We use

$$\cos \theta = \frac{x}{(x^2 + y^2)^{1/2}}$$

and get

$$dE_x = dE \cos \theta = \frac{Q x dy}{8a\pi\epsilon_0(x^2 + y^2)^{3/2}}$$

Now we only need to integrate the  $dE_x$  components to get the total x component of the E-field at point P. We integrate from  $y = -a$  to  $y = a$  with constant x

$$E_x = \int_{y=-a}^{y=a} dE_x = \frac{Q x}{8a\pi\epsilon_0} \int_{y=-a}^{y=a} \frac{dy}{(x^2 + y^2)^{3/2}}$$

The integral may be looked up in a table:

$$\int_{-a}^a \frac{dy}{(x^2 + y^2)^{3/2}} = \left[ \frac{y}{x^2(x^2 + y^2)^{1/2}} \right]_{y=-a}^{y=a} = \frac{a}{x^2(x^2 + a^2)^{1/2}} - \frac{-a}{x^2(x^2 + a^2)^{1/2}} = \frac{2a}{x^2(x^2 + a^2)^{1/2}}$$

Now we get:

$$E_x = \int_{y=-a}^{y=a} dE_x = \frac{Q x}{8a\pi\epsilon_0} \int_{y=-a}^{y=a} \frac{dy}{(x^2 + y^2)^{3/2}} = \frac{Q x}{8a\pi\epsilon_0} \frac{2a}{x^2(x^2 + a^2)^{1/2}} \Rightarrow$$

$$E_x = \frac{Q}{4\pi\epsilon_0 x(x^2 + a^2)^{1/2}} \quad \text{eq. 2.7.1}$$

If we are close to the line of charge or if we have a long line of charge, then  $x \ll a$ .

$$E_x = \frac{Q}{4\pi\epsilon_0 x(a^2)^{1/2}} = \frac{Q}{4\pi\epsilon_0 x a} = \frac{Q}{2\pi\epsilon_0 x 2a} = \frac{\lambda}{2\pi\epsilon_0 x} \quad \text{eq. 2.7.2}$$

where we have inserted  $\lambda = Q/2a$  in the last step.

That means: If we have a very long line of charge, or if we are close to the line of charge, E is perpendicular to the line and depends on  $1/x$  rather than  $1/x^2$ .

## 2.8 The E-field from a ring of charge

A ring-shaped conductor carries a total charge  $dQ$ , see fig.2.8.1. The reason for using  $dQ$  for the total charge of the ring is, that this ring becomes a part of a disc in the next section (2.9). The total charge of the disc is  $Q$  as shown in fig.2.9.1. We want to find the electric field at a point  $P$  on the axis of the ring.

As a start we calculate the field  $dE$  from the small point charge  $dq$ , then we take the sum of all  $dE$  and get  $E$ .

If we take two segments at the top (red charge) and at the bottom of the ring (yellow charge), we can see, that their  $dE_y$  components are in opposite directions. Hence, all net contributions will be in the  $x$  direction.

$$dE = \frac{dq}{4\pi\epsilon_0(x^2 + r^2)}$$

Since

$$\cos(\theta) = \frac{x}{(x^2 + r^2)^{1/2}}$$

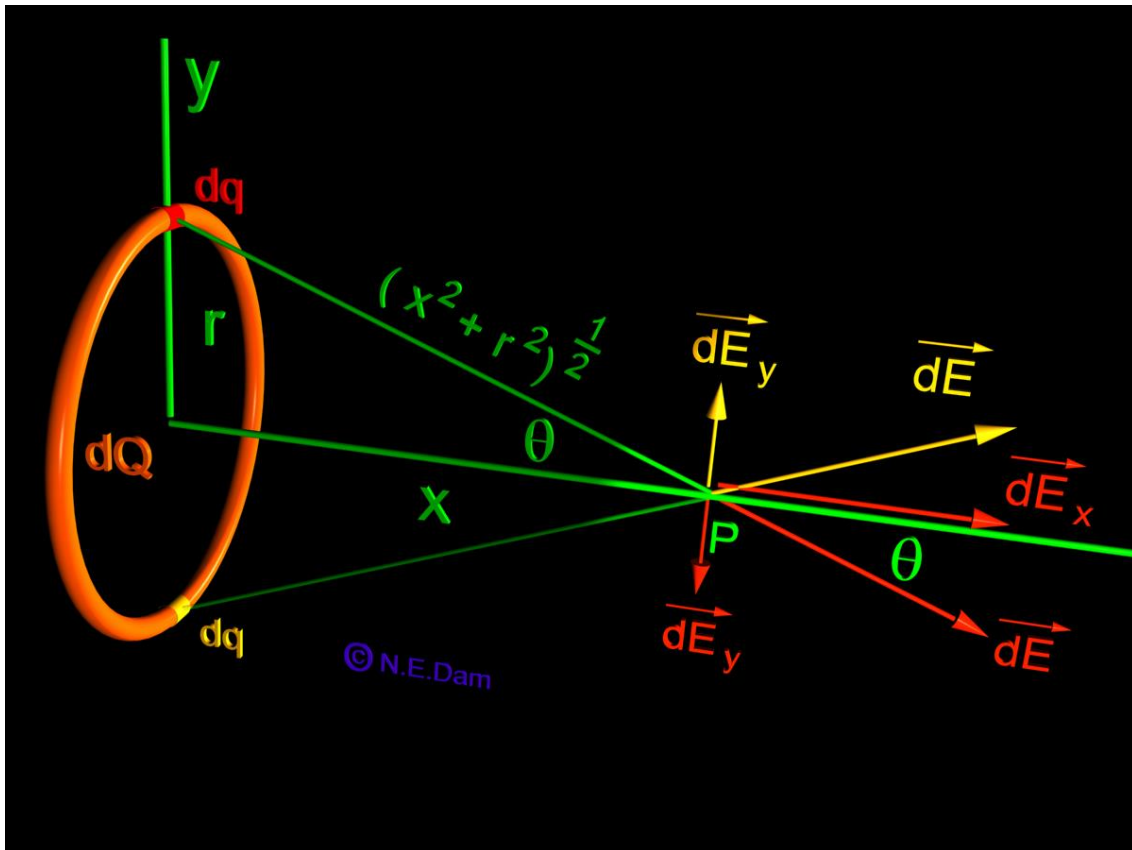


Fig.2.8.1, (N.E. Dam ©). Calculation of the E-field from a ring of charge.

we have for the x component of dE

$$dE_x = \frac{x \, dq}{4\pi\epsilon_0(x^2 + r^2)^{3/2}}$$

Then

$$E_x = \oint_{\text{Ring}} dE_x = \oint_{\text{Ring}} \frac{x \, dq}{4\pi\epsilon_0(x^2 + r^2)^{3/2}} = \frac{x}{4\pi\epsilon_0(x^2 + r^2)^{3/2}} \oint_{\text{Ring}} dq$$

In this integration x and r are constant, so the integration is only for the charge.

$$E_x = \frac{x \, dQ}{4\pi\epsilon_0(x^2 + r^2)^{3/2}} \quad \text{eq. 2.8.1}$$

and  $E_y = 0$

When  $x = 0$  we have  $E_x = 0$ .

When  $x \rightarrow +\infty$  we have

$$E_x = \frac{x \, dQ}{4\pi\epsilon_0(x^2)^{3/2}} = \frac{dQ}{4\pi\epsilon_0(x^2)} \quad \text{eq. 2.8.2}$$

It means that the ring acts like a point charge at long distances.

## 2.9 The E-field from a disc of charge

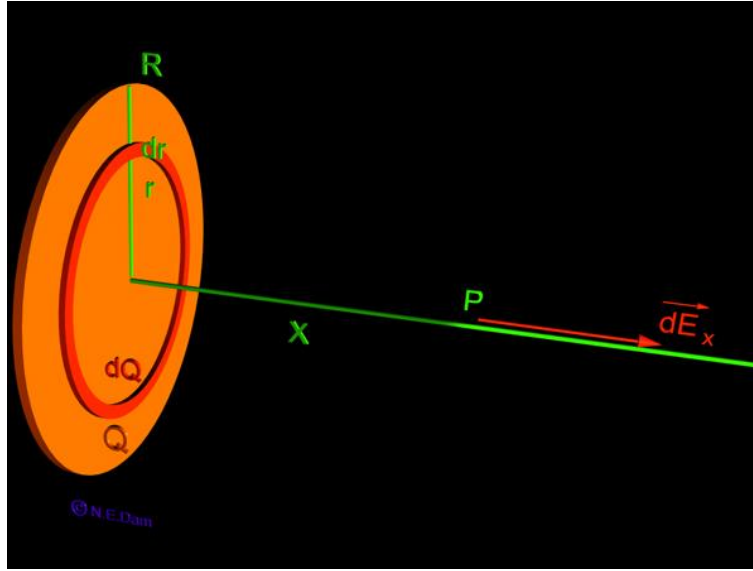


Fig.2.9.1, (N.E. Dam ©). Calculation of the E-field from a disc of charge.

We want to find the electric field caused by a disc of radius  $R$ . The disc is placed with its centre at  $(0,0,0)$ . We define a  $x$ -axis through  $(0,0,0)$  and perpendicular to the disc. The field point, where we want to find the E-field, is on the  $x$ -axis at point  $P$  in distance  $x$  from the disc centre as shown in fig.2.9.1.

The disc has a uniform positive surface charge density:

$$\sigma = \frac{Q}{\pi R^2} \quad \text{eq. 2.9.1}$$

We can represent the charge distribution as a collection of rings of charge  $dQ$  (calculated in section 2.8), and just add up the contributions.

Now we consider the red-orange ring shown in fig.2.9.1. The radius is  $r$  and the circumference of the ring is  $2\pi r$ . The width of the ring is  $dr$ . Hence the area of the ring is  $2\pi r dr$ . The charge on the ring is then  $dQ = \sigma 2\pi r dr$ . This charge is substituted instead of  $dQ$  into eq.2.8.1. The E-field from the ring is now denoted  $dE_x$  instead of  $E_x$

$$dE_x = \frac{x dQ}{4\pi\epsilon_0(x^2 + r^2)^{3/2}} = \frac{x (\sigma 2\pi r dr)}{4\pi\epsilon_0(x^2 + r^2)^{3/2}} \quad \text{eq. 2.9.2}$$

Now, we integrate from  $r = 0$  to  $r = R$  and find

$$E_x = \int_{r=0}^{r=R} dE_x = \frac{\sigma}{2\epsilon_0} \left\{ 1 - \frac{1}{\left\{ 1 + \frac{R^2}{x^2} \right\}^{1/2}} \right\} \quad \text{eq. 2.9.3}$$

If the field point  $P$  is close to the disc, we get the important result for the E-field close to a large plane surface

$$E_x = \frac{\sigma}{2\epsilon_0} \left\{ 1 - \frac{1}{\left\{ 1 + \frac{R^2}{x^2} \right\}^{1/2}} \right\} \rightarrow \frac{\sigma}{2\epsilon_0} \quad \text{when } x \rightarrow 0 \quad \text{eq. 2.9.4}$$

### Derivation of equation 2.9.3

We start from equation 2.9.2

$$dE_x = \frac{x dQ}{4\pi\epsilon_0(x^2 + r^2)^{3/2}} = \frac{x (\sigma 2\pi r dr)}{4\pi\epsilon_0(x^2 + r^2)^{3/2}} \quad \text{eq. 2.9.2}$$

We find

$$dE_x = \frac{\sigma}{2\epsilon_0} \frac{x (r dr)}{(x^2 + r^2)^{3/2}} = \frac{\sigma}{2\epsilon_0} \frac{x (r dr)}{\left\{x^2 \left(1 + \frac{r^2}{x^2}\right)\right\}^{3/2}} \quad \text{eq. 2.9.5}$$

We continue

$$dE_x = \frac{\sigma}{2\epsilon_0} \frac{x (r dr)}{\{x^2\}^{3/2} \left\{1 + \frac{r^2}{x^2}\right\}^{3/2}} = \frac{\sigma}{2\epsilon_0} \frac{r dr}{x^2 \left\{1 + \frac{r^2}{x^2}\right\}^{3/2}} \quad \text{eq. 2.9.6}$$

Now, we should perform the following integration

$$E_x = \int_{r=0}^{r=R} dE_x = \frac{\sigma}{2\epsilon_0} \int_0^R \frac{r dr}{x^2 \left\{1 + \frac{r^2}{x^2}\right\}^{3/2}} \quad \text{eq. 2.9.7}$$

We know that  $r^{-1/2}$  differentiates into  $\frac{-1}{2}r^{-3/2}$  and  $-r^{-1/2}$  into  $\frac{1}{2}r^{-3/2}$ .

Therefore, we guess that the integral could be

$$-\left\{1 + \frac{r^2}{x^2}\right\}^{-1/2}$$

We differentiate this function and find

$$\frac{d}{dr} \left[ -\left\{1 + \frac{r^2}{x^2}\right\}^{-1/2} \right] = \frac{1}{2} \left\{1 + \frac{r^2}{x^2}\right\}^{-3/2} \frac{2r dr}{x^2} = \left\{1 + \frac{r^2}{x^2}\right\}^{-3/2} \frac{r dr}{x^2}$$

The integration test shows that we have found the right integral.

Now, we may calculate the final result

$$E_x = \frac{\sigma}{2\epsilon_0} \int_0^R \frac{r dr}{x^2 \left\{1 + \frac{r^2}{x^2}\right\}^{3/2}} = \frac{\sigma}{2\epsilon_0} \left[ -\left\{1 + \frac{r^2}{x^2}\right\}^{-1/2} \right]_0^R \quad \text{eq. 2.9.8}$$

We swap the limits and get rid of the minus sign

$$E_x = \frac{\sigma}{2\epsilon_0} \left[ \left\{1 + \frac{r^2}{x^2}\right\}^{-1/2} \right]_R^0 = \frac{\sigma}{2\epsilon_0} \left[ \frac{1}{\left\{1 + \frac{r^2}{x^2}\right\}^{1/2}} \right]_R^0 = \frac{\sigma}{2\epsilon_0} \left\{ 1 - \frac{1}{\left\{1 + \frac{R^2}{x^2}\right\}^{1/2}} \right\} \quad \text{eq. 2.9.9}$$

## 2.10 The electric dipole

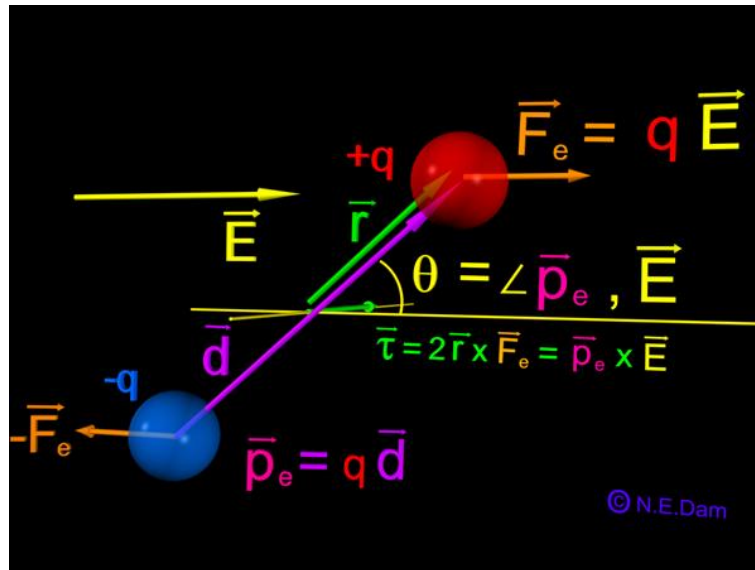


Fig.2.10.1, (N.E. Dam ©). Derivation of the torque for an electric dipole in an E-field.

Fig.2.10.1 shows an electric dipole, which is defined as two charges  $q$  ( $>0$ ) and  $-q$  ( $<0$ ) separated at distance  $\vec{d}$  (from  $-q$  to  $q$ ).

The dipole is situated in a uniform electric field  $\vec{E}$ . The total force on the dipole is zero. As it might be seen from the figure, there will be a torque  $\vec{\tau}$  trying to rotate the dipole to be parallel with the electric field  $\vec{E}$ . The positive charge will be displaced in direction of  $\vec{E}$ , while the negative charge will be displaced in the opposite direction of  $\vec{E}$ .

*The electric dipole moment* is defined as

$$\vec{p}_e = q \vec{d} = q 2\vec{r} \quad \text{eq. 2.10.1}$$

*The torque vector* is then

$$\begin{aligned} \vec{\tau} &= 2\vec{r} \times \vec{F}_e = \frac{\vec{p}_e}{q} \times q\vec{E} \Rightarrow \\ \vec{\tau} &= \vec{p}_e \times \vec{E} \quad \text{eq. 2.10.2} \end{aligned}$$

The factor 2 is because the charge  $-q$  gives the same contribution to the torque as  $+q$ . The force on  $-q$  has the same magnitude as the force on  $+q$ , but the two forces are in opposite directions. Therefore the total force on the dipole is zero, but the torque is not zero, since the forces on  $+q$  and  $-q$  contribute to the torque in the same direction.

When the dipole has been rotated to be parallel with  $\vec{E}$ , the torque will be zero. Then the angle  $\theta$  from  $\vec{p}_e$  to  $\vec{E}$  is zero.

The maximum torque is when the dipole is perpendicular to the E-field, i.e. when  $\theta = 90^\circ$ .

When  $\theta = 180^\circ$  the dipole is unstable. The torque is zero because  $\vec{p}_e$  is (anti) parallel with  $\vec{E}$ . However, if we push a little bit to the dipole so  $\theta$  becomes a little bit less than  $180^\circ$  the torque is not zero any more and the torque will rotate the dipole  $180^\circ$  back to the stable position at  $\theta = 0^\circ$ . If we push to dipole in the stable position, then the dipole rotates back again to the stable position.

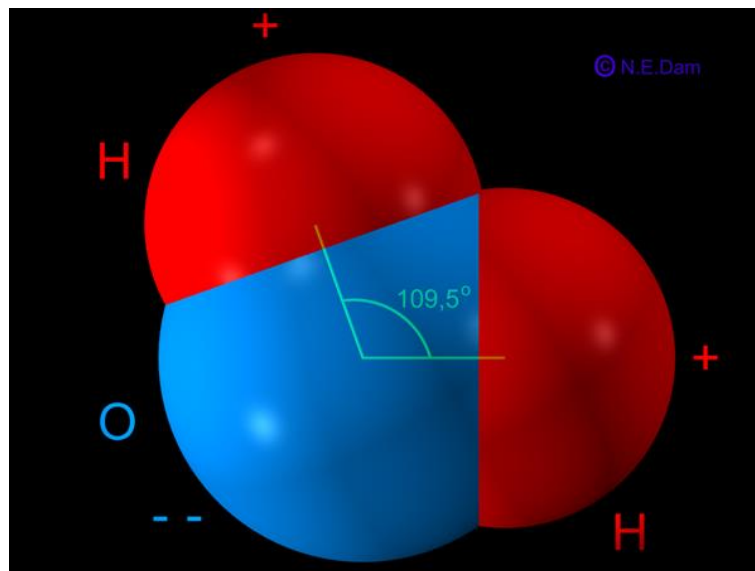


*Fig.2.10.2, (N.E. Dam ©). A water molecule is an electric dipole.*

Fig.2.10.2 shows that a water molecule is an electric dipole. The blue oxygen atom (O) gains an electron from each of the two red hydrogen (H) atoms. Then O has a complete outer shell with 8 electrons and both H atoms have lost their electron, so they have no electrons any more. Thereby we have localized positive charge at the H atoms (or  $H^+$  ions) and localized negative charge at the O atom (or  $O^{2-}$  ions).

## Problems

### 2.1 Vector addition of electric forces along a line

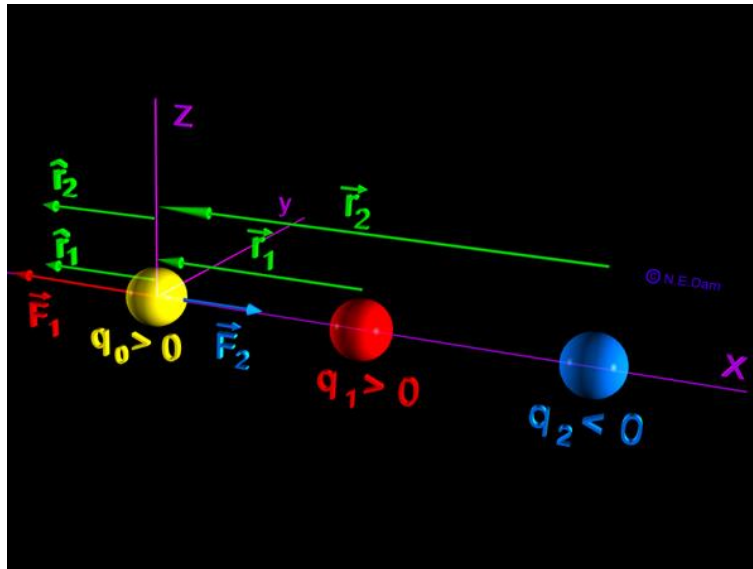


Fig.2.1, (N.E. Dam ©). Vector addition of electric forces along a line.

Fig.2.1 shows  $q_0$  and two other charges  $q_1$  and  $q_2$ , all three charges are placed along a line.

Find the total vector force exerted by  $q_1$  and  $q_2$  on  $q_0$  when  $q_0 = 1 \text{ pC}$ ,  $q_1 = 1 \text{ nC}$ ,  $q_2 = -1 \text{ nC}$ ,  $r_1 = 1 \text{ cm}$ ,  $r_2 = 2 \text{ cm}$ .

Hint:

We let the line be along the x-axis in a xyz coordinate system, where  $q_0$  is placed at the origin. According to the figure we then have

$$\hat{r}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = (-1, 0, 0) \text{ and } \vec{F}_1 = k_e \frac{q_1 \cdot q_0}{r_1^2} \hat{r}_1$$

$\vec{F}_1$  is directed to the left on the x-axis.

$\vec{F}_2$  is directed to the right on the x-axis.

### 2.2 Vector addition of electric forces in a plane

Fig.2.2 shows  $q_0$  and two other charges  $q_1$  and  $q_2$ , all three charges are placed in the same plane (the xy plane in a xyz coordinate system, where  $q_0$  is placed at the origin).

Find the total vector force exerted by  $q_1$  and  $q_2$  on  $q_0$  when

$q_0 = 1 \text{ pC}$ ,  $q_1 = 1 \text{ nC}$ ,  $q_2 = -1 \text{ nC}$ ,

The position of  $q_1$  (in cm) is  $(2, 6, -1, 0)$

The position of  $q_2$  (in cm) is  $(2, 6, 1, 0)$

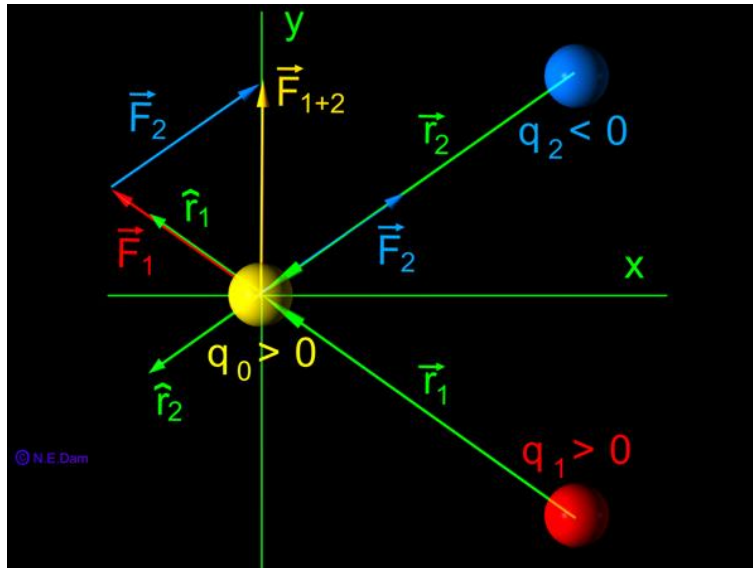


Fig.2.2, (N.E. Dam ©). Vector addition of electric forces in a plane.

### 2.3 Calculation of the E-field from two charges

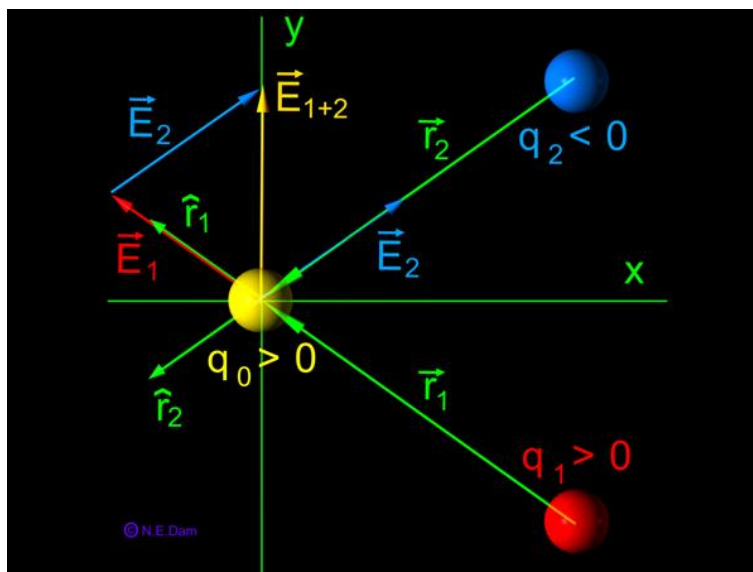


Fig.2.3, (N.E. Dam ©). Calculation of the E-field from two charges

Fig.2.3 shows the same example as used in problem 2.2, see also fig.2.2, where we calculated the total electric force on  $q_0$ . The field point is at P (0,0,0).

1. Use eq.2.5.1 and the total force calculated in problem 2.2 to calculate the total E-field (from  $q_1$  and  $q_2$ ) at point P.
2. Use eq.2.5.1 to calculate the E-field  $\vec{E}_1$  from  $q_1$  (when  $q_2$  is “turned off”) and then calculate the E-field  $\vec{E}_2$  from  $q_2$  (when  $q_1$  is “turned off”). Finally add the two E-fields and find the total E-field using eq.2.7.1. All E-fields should be calculated at point P.
3. Please compare the total E-field from (1) with total E-field from (2). The two results should be equal.

## Test 2<sup>1</sup>

Keywords/questions:

1. 20p Please explain Coulomb's law
2. 10p Definition of the electric field
3. 10p Electric force expressed by the E-field and the charge
4. 10p E-field from a point charge  $q$
5. 10p How can you see the E-field magnitude from a pattern of E-field lines ?
6. 10p How can you see the E-field direction from a pattern of E-field lines ?
7. 10p What is an electric dipole ?
8. 10p Electric dipole moment of an electric dipole (definition)
9. 10p Torque of an electric dipole (derivation)

Total possible maximum points is  $\Sigma$  100p.

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<sup>1</sup> NB. The following are only examples of questions. The real test may differ from this. The evaluation of the test, which is handed in, reflects not only the correctness of the answer but also the care and dedication which has been put into the answer.

### General guidelines for answering the test

The test is without any aids, only pencils and rulers etc. are permitted.

NB. Coloured pencils should be used when drawing sketches and diagrams to add clarity.

Very often, the "questions" are not formulated as precise questions but only as keywords.

Eg., if the keyword is Coulomb's law, you are supposed to do the following:

- a. Write the formula correctly
- b. Explain the symbols in the formula briefly
- c. Make a small coloured sketch
- d. Answer in a well structured, orderly manner
- e. Have a hand-writing which is easy to read

If you just write the formula correctly (a) without explaining the symbols, the evaluation will only be average. If you explain the symbols in the formula briefly (b), the evaluation will be higher. Furthermore, by including a small coloured sketch to the answer (c), almost the highest evaluation may be attained. Not all questions require a sketch, perhaps 2-3 questions per test. It will be indicated when a sketch or a derivation of a formula is desirable. The answer should be well structured so that it is easy to correct. Similarly, the hand writing should be clearly legible, which is not necessarily the same as being "nice".